Chapter 12

Exercise 12.1

1 **a)** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A, y_B - y_A, z_B - z_A) = \left(1 - \left(-\frac{3}{2}\right), -\frac{5}{2} - \left(-\frac{1}{2}\right), 1 - 1\right) = \left(\frac{5}{2}, -2, 0\right)$

b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A, y_B - y_A, z_B - z_A) = \left(1 - (-2), \sqrt{3} - (-\sqrt{3}), -\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = (3, 2\sqrt{3}, 0)$

c) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A, y_B - y_A, z_B - z_A) = (1 - 2, -1 - (-3), 3 - 5) = (-1, 2, -2)$

d) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A, y_B - y_A, z_B - z_A) = (-a - a, -2a - (-a), a - 2a) = (-2a, -a, -a)$

2 **a)** Given $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_Q - x_P, y_Q - y_P, z_Q - z_P) = \left(1, -\frac{5}{2}, 1\right)$

$\Rightarrow \left(x_Q - \left(-\frac{3}{2}\right), y_Q - \left(-\frac{1}{2}\right), z_Q - 1\right) = \left(1, -\frac{5}{2}, 1\right)$. Therefore:

$x_Q - \left(-\frac{3}{2}\right) = 1 \Rightarrow x_Q = 1 - \frac{3}{2} = -\frac{1}{2}$

$y_Q - \left(-\frac{1}{2}\right) = -\frac{5}{2} \Rightarrow y_Q = -\frac{5}{2} - \frac{1}{2} = -3$

$z_Q - 1 = 1 \Rightarrow z_Q = 1 + 1 = 2$. So, $Q\left(-\frac{1}{2}, -3, 2\right)$.

b) Given $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_Q - x_P, y_Q - y_P, z_Q - z_P) = \left(-\frac{3}{2}, -\frac{1}{2}, 1\right)$

$\Rightarrow \left(1 - x_P, -\frac{5}{2} - y_P, 1 - z_P\right) = \left(-\frac{3}{2}, -\frac{1}{2}, 1\right)$. Therefore:

$1 - x_P = -\frac{3}{2} \Rightarrow x_P = 1 + \frac{3}{2} = \frac{5}{2}$

$-\frac{5}{2} - y_P = -\frac{1}{2} \Rightarrow y_P = -\frac{5}{2} + \frac{1}{2} = -2$

$1 - z_P = 1 \Rightarrow z_P = 1 - 1 = 0$. So, $P\left(\frac{5}{2}, -2, 0\right)$.

c) Given $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_Q - x_P, y_Q - y_P, z_Q - z_P) = (-a, -2a, a)$

$\Rightarrow (x_Q - a, y_Q - (-2a), z_Q - 2a) = (-1, -2a, a)$. Therefore:

$x_Q - a = -a \Rightarrow x_Q = 0$

$y_Q - (-2a) = -2a \Rightarrow y_Q = -2a - 2a = -4a$

$z_Q - 2a = a \Rightarrow z_Q = a + 2a = 3a$. So, $Q(0, -4a, 3a)$.

3 **a)** For points M, A and B to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$. If the two vectors are parallel, then one of them is a scalar multiple of the other, for example, $\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$\overrightarrow{AM} = (x_M - x_A, y_M - y_A, z_M - z_A) = (x - 0, y - 0, z - 5)$

$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A) = (1 - 0, 1 - 0, 0 - 5)$

Therefore: $(x, y, z-5) = t(1, 1, -5) \Rightarrow (x, y, z-5) = (t, t, -5t) \Rightarrow x = t,\ y = t,\ z-5 = -5t$

So, $M(t, t, 5-5t)$, where $t \in \mathbb{R}$.

Note: If we start with the condition $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will have:

$\overrightarrow{BM} = (x_M - x_B, y_M - y_B, z_M - z_b) = (x-1, y-1, z-0)$, $\overrightarrow{AB} = (1, 1, -5)$

Therefore: $(x-1, y-1, z) = (t, t, -5t) \Rightarrow x = 1+t,\ y = 1+t,\ z = -5t$

So, $M(1+t, 1+t, -5t)$, where $t \in \mathbb{R}$.

Both conditions describe the same set of points; for example, we can obtain point $M(0, 0, 5)$ by putting $t = 0$ in $M(t, t, 5-5t)$, or $t = -1$ in $M(1+t, 1+t, -5t)$; or $M(2, 2, -5)$ by putting $t = 2$ in $M(t, t, 5-5t)$, or $t = 1$ in $M(1+t, 1+t, -5t)$.

b) For points M, A and B to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$. So, let's say,

$\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$\overrightarrow{AM} = (x_M - x_A, y_M - y_A, z_M - z_A) = (x-(-1), y-0, z-1)$

$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A) = (3-(-1), 5-0, -2-1)$

Therefore: $(x+1, y, z-1) = t(4, 5, -3) \Rightarrow (x+1, y, z-1) = (4t, 5t, -3t) \Rightarrow x = -1+4t,\ y = 5t,\ z = 1-3t$
So, $M(-1+4t, 5t, 1-3t)$, where $t \in \mathbb{R}$.

Note: If we start with the condition $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will have $\overrightarrow{BM} = (x-3, y-5, z+2)$;

therefore, from $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will find $x = 3+4t,\ y = 5+5t,\ z = -2-3t$.

c) For points M, A and B to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$. So, let's say,

$\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$\overrightarrow{AM} = (x_M - x_A, y_M - y_A, z_M - z_A) = (x-2, y-3, z-4)$

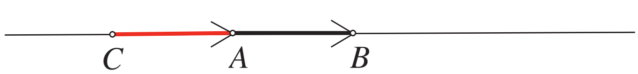
$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A) = (-2-2, -3-3, 5-4)$

Therefore:
$(x-2, y-3, z-4) = t(-4, -6, 1) \Rightarrow (x-2, y-3, z-4) = (-4t, -6t, t) \Rightarrow x = 2-4t,\ y = 3-6t,\ z = 4+t$
So, $M(2-4t, 3-6t, 4+t)$, where $t \in \mathbb{R}$.

Note: If we start with the condition $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will have $\overrightarrow{BM} = (x+2, y+3, z-5)$;

therefore, from $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will find $x = -2-4t,\ y = -3-6t,\ z = 5+t$.

4 If C is the symmetric image of B with respect to A, points A, B and C are on the line and their positions are as shown below. Their relationship can be expressed using different vector relationships; for example, $\overrightarrow{AB} = \overrightarrow{CA}$, $\overrightarrow{AB} = -\overrightarrow{AC}$, $\overrightarrow{CB} = 2\,\overrightarrow{AB}$, ... Here, we will use $\overrightarrow{AB} = \overrightarrow{CA}$.



a) For $C(x, y, z)$:

$\overrightarrow{AB} = (-1-3, 0-(-4), 1-0) = (-4, 4, 1)$, $\overrightarrow{CA} = (3-x, -4-y, 0-z)$. Therefore:

$(3-x, -4-y, 0-z) = (-4, 4, 1) \Rightarrow 3+4 = x,\ -4-4 = y,\ z = -1$; so, $C(7, -8, -1)$.

b) For $C(x, y, z)$:

$\overrightarrow{AB} = \left(-1-(-1), \frac{1}{2}-3, \frac{1}{3}-5\right) = \left(0, -\frac{5}{2}, -\frac{14}{3}\right)$, $\overrightarrow{CA} = (-1-x, 3-y, 5-z)$. Therefore:

$\left(0, -\frac{5}{2}, -\frac{14}{3}\right) = (-1-x, 3-y, 5-z) \Rightarrow x = -1, y = 3+\frac{5}{2} = \frac{11}{2}, z = 5+\frac{14}{3} = \frac{29}{3}$; so,

$C\left(-1, \frac{11}{2}, \frac{29}{3}\right)$.

c) For $C(x, y, z)$:

$\overrightarrow{AB} = (a-1, 2a-2, b-(-1))$, $\overrightarrow{CA} = (1-x, 2-y, -1-z)$. Therefore:

$(a-1, 2a-2, b+1) = (1-x, 2-y, -1-z)$

$\Rightarrow x = 1-a+1 = 2-a, y = 2-2a+2 = 4-2a, z = -1-b-1 = -2-b$; so,

$C(2-a, 4-2a, -2-b)$.

5 **a)** For $G(x, y, z)$:

$\vec{0} = \overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC} = (-1-x, -1-y, -1-z)+(-1-x, 2-y, -1-z)+(1-x, 2-y, 3-z)$

$\Rightarrow (0,0,0) = (-1-3x, 3-3y, 1-3z)$

$\Rightarrow x = -\frac{1}{3}, y = 1, z = \frac{1}{3}$. So, $G\left(-\frac{1}{3}, 1, \frac{1}{3}\right)$.

b) For $G(x, y, z)$:

$\vec{0} = \overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC} = (2-x, -3-y, 1-z)+(1-x, -2-y, -5-z)+(0-x, 0-y, 1-z)$

$\Rightarrow (0,0,0) = (3-3x, -5-3y, -3-3z)$

$\Rightarrow x = 1, y = -\frac{5}{3}, z = -1$. So, $G\left(1, -\frac{5}{3}, -1\right)$.

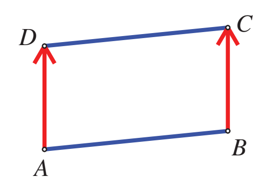
c) For $G(x, y, z)$:

$\vec{0} = \overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC} = (a-x, 2a-y, 3a-z)+(b-x, 2b-y, 3b-z)+(c-x, 2c-y, 3c-z)$

$\Rightarrow (0,0,0) = (a+b+c-3x, 2a+2b+2c-3y, 3a-3b-3c-3z)$

$\Rightarrow x = \frac{a+b+c}{3}, y = \frac{2a+2b+2c}{3}, z = a+b+c$. So, $G\left(\frac{a+b+c}{3}, \frac{2a+2b+2c}{3}, a+b+c\right)$.

6 The relationship between points A, B, C, and D of parallelogram $ABCD$ can be expressed using different vector relationships; for example, $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$, $\overrightarrow{BA} = \overrightarrow{CD}$, ... Here, we will use $\overrightarrow{AD} = \overrightarrow{BC}$.



a) For $D(x, y, z)$:

$\overrightarrow{BC} = \left(-\sqrt{3}-1, 2-3, -5-0\right) = \left(-\sqrt{3}-1, -1, -5\right)$

$\overrightarrow{AD} = \left(x-\sqrt{3}, y-2, z-(-1)\right) = \left(x-\sqrt{3}, y-2, z+1\right)$

Therefore: $-\sqrt{3}-1 = x-\sqrt{3} \Rightarrow x = -1$, $y-2 = -1 \Rightarrow y = 1$, $z+1 = -5 \Rightarrow z = -6$

So, $D(-1, 1, -6)$.

b) For $D(x, y, z)$:

$\overrightarrow{BC} = \left(-2\sqrt{2}-3\sqrt{2}, \sqrt{3}-\left(-\sqrt{3}\right), -3\sqrt{5}-5\sqrt{5}\right) = \left(-5\sqrt{2}, 2\sqrt{3}, -8\sqrt{5}\right)$

$\overrightarrow{AD} = \left(x-\sqrt{2}, y-\sqrt{3}, z-\sqrt{5}\right)$

Therefore: $-5\sqrt{2} = x - \sqrt{2} \Rightarrow x = -4\sqrt{2}$, $y - \sqrt{3} = 2\sqrt{3} \Rightarrow y = 3\sqrt{3}$, $z - \sqrt{5} = -8\sqrt{5} \Rightarrow z = -7\sqrt{5}$

So, $D\left(-4\sqrt{2}, 3\sqrt{3}, -7\sqrt{5}\right)$.

c) For $D(x, y, z)$:

$$\overrightarrow{BC} = \left(\frac{7}{2} - \frac{1}{2}, -\frac{1}{3} - \frac{2}{3}, 1 - 5\right) = (3, -1, -4)$$

$$\overrightarrow{AD} = \left(x + \frac{1}{2}, y - \frac{1}{3}, z - 0\right) = \left(x + \frac{1}{2}, y - \frac{1}{3}, z\right)$$

Therefore: $3 = x + \frac{1}{2} \Rightarrow x = \frac{5}{2}$, $y - \frac{1}{3} = -1 \Rightarrow y = -\frac{2}{3}$, $z = -4$

So, $D\left(\frac{5}{2}, -\frac{2}{3}, -4\right)$.

7 Two vectors **v** and **w** have the same direction if, for $t > 0$, $\mathbf{v} = t\mathbf{w}$. Therefore:

$(m - 2, m + n, -2m + n) = t(2, 4, -6)$.

$$\begin{cases} m - 2 = 2t \\ m + n = 4t \\ -2m + n = -6t \end{cases}$$

$$\Rightarrow \begin{cases} m = 2t + 2 \\ m + n = 4t \\ -2m + n = -6t \end{cases} \Rightarrow \begin{matrix} 2t + 2 + n = 4t \Rightarrow n = 2t - 2 \\ -2m + n = -6t \end{matrix} \Rightarrow -2(2t + 2) + (2t - 2) = -6t \Rightarrow t = \frac{3}{2}$$

Therefore: $m = 2t + 2 = 5$, $n = 2t - 2 = 1$.

8 **a)** The length of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is $\sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$, so the unit vector is $\frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$.

b) The length of the vector $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ is $\sqrt{6^2 + (-4)^2 + 2^2} = \sqrt{56} = 2\sqrt{14}$, so the unit vector is $\frac{1}{2\sqrt{14}}(6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = \frac{3}{\sqrt{14}}\mathbf{i} - \frac{2}{\sqrt{14}}\mathbf{j} + \frac{1}{\sqrt{14}}\mathbf{k}$.

c) The length of the vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is $\sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$, so the unit vector is $\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$.

9 **a)** $\mathbf{u} + \mathbf{v} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + (2\mathbf{i} + \mathbf{j}) = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$|\mathbf{u} + \mathbf{v}| = |3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{29}$

b) $|\mathbf{u}| + |\mathbf{v}| = |\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}| + |2\mathbf{i} + \mathbf{j}| = \sqrt{1^2 + 3^2 + (-2)^2} + \sqrt{2^2 + 1^2 + 0^2} = \sqrt{14} + \sqrt{5}$

c) $-3\mathbf{u} = -3(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$; $3\mathbf{v} = 3(2\mathbf{i} + \mathbf{j}) = 6\mathbf{i} + 3\mathbf{j}$

$|-3\mathbf{u}| + |3\mathbf{v}| = \sqrt{(-3)^2 + (-9)^2 + 6^2} + \sqrt{6^2 + 3^2 + 0^2} = \sqrt{126} + \sqrt{45} = 3\sqrt{14} + 3\sqrt{5}$

d) $\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{\sqrt{14}}\mathbf{u} = \frac{1}{\sqrt{14}}(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{3}{\sqrt{14}}\mathbf{j} - \frac{2}{\sqrt{14}}\mathbf{k}$

e) $\left|\frac{1}{|\mathbf{u}|}\mathbf{u}\right| = \left|\frac{1}{\sqrt{14}}\mathbf{i} + \frac{3}{\sqrt{14}}\mathbf{j} - \frac{2}{\sqrt{14}}\mathbf{k}\right| = \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2 + \left(\frac{-2}{\sqrt{14}}\right)^2} = \sqrt{\frac{1 + 9 + 4}{14}} = 1$

10 **a)** Using $B(x, y, z)$ for the terminal point and $A(-1, 2, -3)$ for the initial point:

$$\overrightarrow{AB} = (x-(-1), y-2, z-(-3)) = (x+1, y-2, z+3), \overrightarrow{AB} = \mathbf{w}.$$

Therefore: $\begin{pmatrix} x+1 \\ y-2 \\ z+3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \Rightarrow x = 3, y = 4, z = -5$. So, the terminal point is $(3, 4, -5)$.

b) Using $B(x, y, z)$ for the terminal point and $A(-2, 1, 4)$ for the initial point:

$$\overrightarrow{AB} = (x-(-2), y-1, z-4) = (x+2, y-1, z-4), \overrightarrow{AB} = \mathbf{v}.$$

Therefore: $\begin{pmatrix} x+2 \\ y-1 \\ z-4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \Rightarrow x = 0, y = -2, z = 5$. So, the terminal point is $(0, -2, 5)$.

11 **a)** A vector opposite in direction and a third the magnitude of $\mathbf{u}$ is $-\frac{1}{3}\mathbf{u}$. Therefore:

$$-\frac{1}{3}\mathbf{u} = -\frac{1}{3}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{4}{3} \end{pmatrix}.$$

b) A vector in the same direction as $\mathbf{w}$ and whose magnitude equals 12 is 12 times a unit vector in the direction of $\mathbf{w}$. Therefore, the vector is of the form:

$$12\frac{1}{|\mathbf{w}|}\mathbf{w} = 12\frac{1}{\sqrt{4^2+2^2+(-2)^2}}(4\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = \frac{12}{\sqrt{24}}(4\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = \sqrt{6}(4\mathbf{i}+2\mathbf{j}-2\mathbf{k}).$$

c) If vectors are parallel, then one can be represented as t times the other. Therefore:

$x\mathbf{i} + y\mathbf{j} - 2\mathbf{k} = t(\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$. From the **k**-coordinate, we can find the value of t: $-2 = 3t \Rightarrow t = -\frac{2}{3}$.

So, $x = -\frac{2}{3}\cdot 1 = -\frac{2}{3}$ and $y = -\frac{2}{3}\cdot(-4) = \frac{8}{3}$, and the vector is: $-\frac{2}{3}\mathbf{i} + \frac{8}{3}\mathbf{j} - 2\mathbf{k}$.

12 Let $\mathbf{u}$ be the vector from the vertex A to the midpoint of side BC; so, $\mathbf{u} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$. Let $\mathbf{v}$ be the vector from the vertex B to the midpoint of side AC; so, $\mathbf{v} = \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$. Let $\mathbf{w}$ be the vector from the vertex C to the midpoint of side AB; so, $\mathbf{w} = \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB}$. Adding the vectors:

$$\mathbf{u}+\mathbf{v}+\mathbf{w} = \cancel{\overrightarrow{AB}} + \frac{1}{2}\overrightarrow{BC} + \cancel{\overrightarrow{BA}} + \frac{1}{2}\overrightarrow{AC} + \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BC} + \left(\frac{1}{2}\overrightarrow{AC} + \overrightarrow{CA}\right) + \frac{1}{2}\overrightarrow{AB}$$

$$= \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}\right) = \frac{1}{2}\cdot\vec{0} = \vec{0}$$

Exercise 12.2

1 **a)** $\mathbf{u}\cdot\mathbf{v} = 3\cdot 2 + (-2)\cdot(-1) + 4\cdot(-6) = -16$

$$\cos\theta = \frac{-16}{\sqrt{3^2+(-2)^2+4^2}\sqrt{2^2+(-1)^2+(-6)^2}} = \frac{-16}{\sqrt{29}\sqrt{41}} \Rightarrow \theta \approx 117.65^\circ$$

b) $\mathbf{u}\cdot\mathbf{v} = 2\cdot(-1) + (-6)\cdot 3 + 0\cdot 5 = -20$

$$\cos\theta = \frac{-20}{\sqrt{2^2+(-6)^2+0^2}\sqrt{(-1)^2+(3)^2+5^2}} = \frac{-20}{\sqrt{40}\sqrt{35}} \Rightarrow \theta \approx 122.31^\circ$$

c) $\mathbf{u}\cdot\mathbf{v} = 3\cdot 5 + (-1)\cdot 2 = 13$

$$\cos\theta = \frac{13}{\sqrt{3^2+(-1)^2}\sqrt{5^2+2^2}} = \frac{13}{\sqrt{10}\sqrt{29}} \Rightarrow \theta \approx 40.24^\circ$$

d) $\mathbf{u}\cdot\mathbf{v} = 1\cdot 0 + (-3)\cdot 5 + 0\cdot 2 = -15$

$$\cos\theta = \frac{-15}{\sqrt{1^2+(-3)^2+0^2}\sqrt{0^2+5^2+(-2)^2}} = \frac{-15}{\sqrt{10}\sqrt{29}} \Rightarrow \theta \approx 151.74^\circ$$

e) $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = 3\cdot 4\cdot\cos\frac{\pi}{3} = 3\cdot 4\cdot\frac{1}{2} = 6$; angle is $\theta = 60^\circ$

f) $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = 3\cdot 4\cdot\cos\frac{2\pi}{3} = 3\cdot 4\cdot\frac{-1}{2} = -6$; angle is $\theta = -120^\circ$

2 **a)** $\mathbf{u}\cdot\mathbf{v} = 2\cdot(-1) + (-6)\cdot 3 + 4\cdot 5 = 0$. The dot product is zero; hence, the vectors are orthogonal.

b) $\mathbf{u}\cdot\mathbf{v} = 3\cdot 5 + (-7)\cdot 2 = 1$. The dot product is positive; hence, the angle is acute.

c) $\mathbf{u}\cdot\mathbf{v} = 1\cdot 0 + (-3)\cdot 6 + 6\cdot 3 = 0$. The dot product is zero; hence, the vectors are orthogonal.

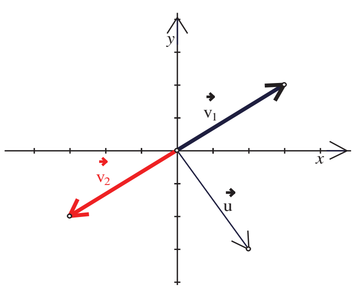
3 **a)** $\mathbf{v}\cdot\mathbf{u} = -y\cdot x + x\cdot y = 0$. The dot product is zero; hence, $\mathbf{v}$ is orthogonal to $\mathbf{u}$.

$\mathbf{w}\cdot\mathbf{u} = y\cdot x + (-x)\cdot y = 0$. The dot product is zero; hence, $\mathbf{w}$ is orthogonal to $\mathbf{u}$.

Pay attention to the relationship between the coordinates of a two-dimensional vector and a vector that is perpendicular to it.

b) The vectors perpendicular to $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ are $3\mathbf{i} + 2\mathbf{j}$ and $-3\mathbf{i} - 2\mathbf{j}$.

Unit vectors in the direction of those vectors are: $\mathbf{v}_1 = \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$ and $\mathbf{v}_2 = -\frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$.



4 **a)** **i)** $|\mathbf{v}| = \sqrt{2^2+(-3)^2+1^2} = \sqrt{14}$

$\mathbf{i}\cdot\mathbf{v} = 1\cdot 2 + 0\cdot(-3) + 0\cdot 1 = 2 \Rightarrow \cos\alpha = \frac{2}{\sqrt{14}}$

$\mathbf{j}\cdot\mathbf{v} = 0\cdot 2 + 1\cdot(-3) + 0\cdot 1 = -3 \Rightarrow \cos\beta = \frac{-3}{\sqrt{14}}$

$\mathbf{k}\cdot\mathbf{v} = 0\cdot 2 + 0\cdot(-3) + 1\cdot 1 = 1 \Rightarrow \cos\gamma = \frac{1}{\sqrt{14}}$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{-3}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 = \frac{4+9+1}{14} = 1$

iii) $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 57.6884...^\circ \approx 58^\circ$, $\beta = \cos^{-1}\left(\frac{-3}{\sqrt{14}}\right) = 143.300\,77...^\circ \approx 143^\circ$,

$\gamma = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.4986...^\circ \approx 74^\circ$

b) **i)** $|\mathbf{v}| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$

$\mathbf{i}\cdot\mathbf{v} = 1\cdot 1 + 0\cdot(-2) + 0\cdot 1 = 1 \Rightarrow \cos\alpha = \frac{1}{\sqrt{6}}$

$\mathbf{j} \cdot \mathbf{v} = 0 \cdot 1 + 1 \cdot (-2) + 0 \cdot 1 = -2 \Rightarrow \cos\beta = \frac{-2}{\sqrt{6}}$

$\mathbf{k} \cdot \mathbf{v} = 0 \cdot 1 + 0 \cdot (-2) + 1 \cdot 1 = 1 \Rightarrow \cos\gamma = \frac{1}{\sqrt{6}}$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{1+4+1}{6} = 1$

iii) $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 65.9051...^\circ \approx 66^\circ, \beta = \cos^{-1}\left(\frac{-2}{\sqrt{6}}\right) = 144.7356...^\circ \approx 145^\circ,$

$\gamma = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 65.9051...^\circ \approx 66^\circ$

c) i) $|\mathbf{v}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\mathbf{i} \cdot \mathbf{v} = 1 \cdot 3 + 0 \cdot (-2) + 0 \cdot 1 = 3 \Rightarrow \cos\alpha = \frac{3}{\sqrt{14}}$

$\mathbf{j} \cdot \mathbf{v} = 0 \cdot 3 + 1 \cdot (-2) + 0 \cdot 1 = -2 \Rightarrow \cos\beta = \frac{-2}{\sqrt{14}}$

$\mathbf{k} \cdot \mathbf{v} = 0 \cdot 3 + 0 \cdot (-2) + 1 \cdot 1 = 1 \Rightarrow \cos\gamma = \frac{1}{\sqrt{14}}$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{3}{\sqrt{14}}\right)^2 + \left(\frac{-2}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 = \frac{9+4+1}{14} = 1$

iii) $\alpha = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 36.6992...^\circ \approx 37^\circ, \beta = \cos^{-1}\left(\frac{-2}{\sqrt{14}}\right) = 122.3115...^\circ \approx 122^\circ,$

$\gamma = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.4986...^\circ \approx 74^\circ$

d) i) $|\mathbf{v}| = \sqrt{3^2 + 0^2 + (-4)^2} = 5$

$\mathbf{i} \cdot \mathbf{v} = 1 \cdot 3 + 0 \cdot 0 + 0 \cdot (-4) = 3 \Rightarrow \cos\alpha = \frac{3}{5}$

$\mathbf{j} \cdot \mathbf{v} = 0 \cdot 3 + 1 \cdot 0 + 0 \cdot (-4) = 0 \Rightarrow \cos\beta = 0$

$\mathbf{k} \cdot \mathbf{v} = 0 \cdot 3 + 0 \cdot 0 + 1 \cdot (-4) = -4 \Rightarrow \cos\gamma = \frac{-4}{5}$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{3}{5}\right)^2 + 0^2 + \left(\frac{-4}{5}\right)^2 = \frac{9+16}{25} = 1$

iii) $\alpha = \cos^{-1}\left(\frac{3}{5}\right) = 53.1301...^\circ \approx 53^\circ, \beta = \cos^{-1} 0 = 90^\circ, \gamma = \cos^{-1}\left(\frac{-4}{5}\right) = 143.1301...^\circ \approx 143^\circ$

5 **a)** $\mathbf{u} \cdot \mathbf{v} = 3 \cdot (m-2) + 5 \cdot (m+3) + 0 \cdot 0 = 8m + 9$

Vectors are perpendicular if their dot product is zero; therefore: $8m + 9 = 0 \Rightarrow m = -\frac{9}{8}$

b) $\mathbf{u} \cdot \mathbf{v} = (2m) \cdot (m-1) + (m-1) \cdot m + (m+1) \cdot (m-1) = 4m^2 - 3m - 1$

Vectors are perpendicular if their dot product is zero; therefore:

$4m^2 - 3m - 1 = 0 \Rightarrow m = 1, m = -\frac{1}{4}$

6 $\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{u} + m\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + m\mathbf{u} \cdot \mathbf{v} = (-3)^2 + 1^2 + 2^2 + m((-3) \cdot 1 + 1 \cdot 2 + 2 \cdot 1) = 14 + m$

Vectors are orthogonal if their dot product is zero; therefore: $14 + m = 0 \Rightarrow m = -14$

7 **a)** $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \dfrac{(-2)\cdot 6 + 5\cdot(-3) + 4\cdot 0}{\sqrt{(-2)^2+5^2+4^2}\sqrt{6^2+(-3)^2+0^2}} = \dfrac{-27}{\sqrt{45}\sqrt{45}} = -\dfrac{27}{45}$

$\Rightarrow \theta = \cos^{-1}\left(-\dfrac{27}{45}\right) = 126.8698...^\circ \approx 127^\circ$

b) $\mathbf{u}+\mathbf{v} = (-2, 5, 4) + (6, -3, 0) = (4, 2, 4)$

$\cos\theta = \dfrac{\mathbf{u}\cdot(\mathbf{u}+\mathbf{v})}{|\mathbf{u}|\,|\mathbf{u}+\mathbf{v}|} = \dfrac{(-2)\cdot 4 + 5\cdot 2 + 4\cdot 4}{\sqrt{45}\sqrt{4^2+2^2+4^2}} = \dfrac{18}{\sqrt{45}\sqrt{36}} = \dfrac{1}{\sqrt{5}}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right) = 63.4349...^\circ \approx 63^\circ$

c) $\cos\theta = \dfrac{\mathbf{v}\cdot(\mathbf{u}+\mathbf{v})}{|\mathbf{v}|\,|\mathbf{u}+\mathbf{v}|} = \dfrac{6\cdot 4 + (-3)\cdot 2 + 0\cdot 4}{\sqrt{45}\cdot 6} = \dfrac{18}{\sqrt{45}\cdot 6} = \dfrac{1}{\sqrt{5}}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{5}}\right) = 63.4349...^\circ \approx 63^\circ$

8 $\overrightarrow{AB} = (3-1, 5-2, -2-(-3)) = (2, 3, 1)$

$\overrightarrow{AC} = (m-1, 1-2, -10m-(-3)) = (m-1, -1, -10m+3)$

a) The points A, B and C are collinear if $\overrightarrow{AC}$ is parallel to $\overrightarrow{AB}$. Given this collinearity, $\overrightarrow{AC} = t\,\overrightarrow{AB} \Rightarrow (2, 3, 1) = t\,(m-1, -1, -10m+3)$. From the second set of coordinates, we can determine t: $3 = -t \Rightarrow t = -3$. Hence: $2 = -3\cdot(m-1) \Rightarrow m = \dfrac{1}{3}$. Checking with the third set of coordinates: $1 = -3\left(-10\cdot\dfrac{1}{3}+3\right) \Rightarrow 1 = -3\cdot\dfrac{-1}{3}$; so, it fits and for $m = \dfrac{1}{3}$ the points are collinear.

b) $\overrightarrow{AC}\cdot\overrightarrow{AB} \Rightarrow 2\cdot(m-1) + 3\cdot(-1) + 1\cdot(-10m+3) = -8m-2$

Vectors are perpendicular if their dot product is zero; therefore:

$\overrightarrow{AC}\cdot\overrightarrow{AB} = 0 \Rightarrow -8m-2 = 0 \Rightarrow m = -\dfrac{1}{4}$.

9 The vector equation of the line is an equation of the form: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0$ is the position vector of any point on the line and the direction vector $\mathbf{v}$ is a vector parallel to the line.

For the median through A, we can take the position vector of point A for $\mathbf{r}_0$ and the vector from A to the midpoint of BC for $\mathbf{v}$. So:

$\mathbf{r}_0 = (4, -2, -1)$, $m_{BC} = \left(\dfrac{3+3}{2}, \dfrac{-5+1}{2}, \dfrac{-1+2}{2}\right) = \left(3, -2, \dfrac{1}{2}\right)$,

$\mathbf{v} = \left(3-4, -2-(-2), \dfrac{1}{2}-(-1)\right) = \left(-1, 0, \dfrac{3}{2}\right)$. Therefore:

$m_A: \ \mathbf{r} = (4, -2, -1) + m\left(-1, 0, \dfrac{3}{2}\right)$.

For the median through B, we can take the position vector of point B for $\mathbf{r}_0$ and the vector from B to the midpoint of AC for $\mathbf{v}$. So:

$\mathbf{r}_0 = (3, -5, -1)$, $m_{AC} = \left(\dfrac{4+3}{2}, \dfrac{-2+1}{2}, \dfrac{-1+2}{2}\right) = \left(\dfrac{7}{2}, -\dfrac{1}{2}, \dfrac{1}{2}\right)$,

$\mathbf{v} = \left(\dfrac{7}{2}-3, -\dfrac{1}{2}-(-5), \dfrac{1}{2}-(-1)\right) = \left(\dfrac{1}{2}, \dfrac{9}{2}, \dfrac{3}{2}\right)$. Therefore:

$m_B: \ \mathbf{r} = (3, -5, -1) + n\left(\dfrac{1}{2}, \dfrac{9}{2}, \dfrac{3}{2}\right)$.

For the median through C, we can take the position vector of point C for $\mathbf{r}_0$ and the vector from C to the midpoint of AB for $\mathbf{v}$. So:

$\mathbf{r}_0 = (3,1,2)$, $m_{AB} = \left(\frac{4+3}{2}, \frac{-2+(-5)}{2}, \frac{-1+(-1)}{2}\right) = \left(\frac{7}{2}, -\frac{7}{2}, -1\right)$,

$\mathbf{v} = \left(\frac{7}{2} - 3, -\frac{7}{2} - 1, -1 - 2\right) = \left(\frac{1}{2}, -\frac{9}{2}, -3\right)$. Therefore:

$m_C : \mathbf{r} = (3,1,2) + k\left(\frac{1}{2}, -\frac{9}{2}, -3\right)$.

The centroid is the point where all the medians meet. We will find the intersection point of two lines, and then check that this point is also on the third line.

If m_A and m_B intersect, then:

$(4,-2,-1) + m\left(-1, 0, \frac{3}{2}\right) = (3,-5,-1) + n\left(\frac{1}{2}, \frac{9}{2}, \frac{3}{2}\right)$. Therefore, we have:

$$\begin{cases} 4 - m = 3 + \frac{1}{2}n \\ -2 + 0m = -5 + \frac{9}{2}n \Rightarrow n = \frac{2}{3} \\ -1 + \frac{3}{2}m = -1 + \frac{3}{2}n \end{cases} \Rightarrow m = \frac{2}{3}$$

Putting $m = n = \frac{2}{3}$ we can see that it fits the equation, so the point of intersection of m_A and m_B is:

$(4,-2,-1) + \frac{2}{3}\left(-1, 0, \frac{3}{2}\right) = \left(\frac{10}{3}, -2, 0\right)$. Now, we have to check that $\left(\frac{10}{3}, -2, 0\right)$ is on the third line as well: $\left(\frac{10}{3}, -2, 0\right) = (3,1,2) + \frac{2}{3}\cdot\left(\frac{1}{2}, -\frac{9}{2}, -3\right)$. Hence, we can see that the centroid is $\left(\frac{10}{3}, -2, 0\right)$.

Note: For the centroid, it holds that: $\left(\frac{10}{3}, -2, 0\right) = \left(\frac{4+3+3}{3}, \frac{-2-5+1}{3}, \frac{-1-1+2}{3}\right)$. The formula $\left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3}, \frac{z_A + z_B + z_C}{3}\right)$ holds, in general, for a triangle with vertices A, B, C.

10 $\overrightarrow{AB} = (-3-1, 2-2, 1-3) = (-4, 0, -2)$; $\left|\overrightarrow{AB}\right| = \sqrt{20}$

$\overrightarrow{AC} = (1-1, -4-2, 3-3) = (0, -6, 0)$; $\left|\overrightarrow{AC}\right| = 6$

$\overrightarrow{AD} = (3-1, 2-2, -3-3) = (2, 0, -6)$; $\left|\overrightarrow{AD}\right| = \sqrt{40}$

$\overrightarrow{BC} = (1-(-3), -4-2, 3-1) = (4, -6, 2)$; $\left|\overrightarrow{BC}\right| = \sqrt{56}$

$\overrightarrow{BD} = (3-(-3), 2-2, -3-1) = (6, 0, -4)$; $\left|\overrightarrow{BD}\right| = \sqrt{52}$

$\overrightarrow{CD} = (3-1, 2-(-4), -3-3) = (2, 6, -6)$; $\left|\overrightarrow{CD}\right| = \sqrt{76}$

We will calculate the angles by finding the dot product and using the cosine angle formula:

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 0 \Rightarrow \text{angle} = 90^\circ$

$\overrightarrow{AB} \cdot \overrightarrow{AD} = -4 \Rightarrow \cos^{-1}\left(\dfrac{-4}{\sqrt{20}\sqrt{40}}\right) = 98.1301...^\circ \Rightarrow \text{angle} = 180^\circ - 98.1301...^\circ \approx 82^\circ$

$\overrightarrow{AB} \cdot \overrightarrow{BD} = -16 \Rightarrow \cos^{-1}\left(\dfrac{-16}{\sqrt{20}\sqrt{52}}\right) = 119.7448...^\circ \Rightarrow \text{angle} = 180^\circ - 119.7448...^\circ \approx 60^\circ$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = -20 \Rightarrow \cos^{-1}\left(\dfrac{-20}{\sqrt{20}\sqrt{56}}\right) = 126.6992...^\circ \Rightarrow \text{angle} = 180^\circ - 126.6992...^\circ \approx 53^\circ$

$\overrightarrow{AC} \cdot \overrightarrow{AD} = 0 \Rightarrow \text{angle} = 90^\circ$

$\overrightarrow{AC} \cdot \overrightarrow{BC} = 36 \Rightarrow \cos^{-1}\left(\dfrac{36}{6\sqrt{56}}\right) = 36.6992...^\circ \Rightarrow \text{angle} \approx 37^\circ$

$\overrightarrow{AC} \cdot \overrightarrow{CD} = -36 \Rightarrow \cos^{-1}\left(\dfrac{-36}{6\sqrt{76}}\right) = 133.4915...^\circ \Rightarrow \text{angle} = 180^\circ - 133.4915...^\circ \approx 47^\circ$

$\overrightarrow{AD} \cdot \overrightarrow{BD} = 36 \Rightarrow \cos^{-1}\left(\dfrac{36}{\sqrt{40}\sqrt{52}}\right) = 37.8749...^\circ \Rightarrow \text{angle} \approx 38^\circ$

$\overrightarrow{AD} \cdot \overrightarrow{CD} = 40 \Rightarrow \cos^{-1}\left(\dfrac{40}{\sqrt{40}\sqrt{76}}\right) = 43.4915...^\circ \Rightarrow \text{angle} \approx 43^\circ$

$\overrightarrow{BC} \cdot \overrightarrow{BD} = 16 \Rightarrow \cos^{-1}\left(\dfrac{16}{\sqrt{56}\sqrt{52}}\right) = 72.7525...^\circ \Rightarrow \text{angle} \approx 73^\circ$

$\overrightarrow{BC} \cdot \overrightarrow{CD} = -40 \Rightarrow \cos^{-1}\left(\dfrac{-40}{\sqrt{56}\sqrt{76}}\right) = 127.8168...^\circ \Rightarrow \text{angle} = 180^\circ - 127.8168...^\circ \approx 52^\circ$

$\overrightarrow{BD} \cdot \overrightarrow{CD} = 36 \Rightarrow \cos^{-1}\left(\dfrac{36}{\sqrt{52}\sqrt{76}}\right) = 55.0643...^\circ \Rightarrow \text{angle} \approx 55^\circ$

11 Total surface area consists of triangles *ABC, ABD, ACD,* and *BCD*:

$A_{ABC} = \dfrac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| \sin 90^\circ \approx \dfrac{1}{2}\sqrt{20} \cdot 6 \cdot 1 = 6\sqrt{5} \approx 13.4164$

(**Note:** $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular.)

$A_{ABD} = \dfrac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right| \sin\theta \approx \dfrac{1}{2}\sqrt{20} \cdot \sqrt{40} \cdot \sin 82^\circ \approx 14.0045$

(**Note:** $\theta = \angle\,\overrightarrow{AB}, \overrightarrow{AD} \approx 82^\circ$)

$A_{ACD} = \dfrac{1}{2}\left|\overrightarrow{AC}\right|\left|\overrightarrow{CD}\right| \sin\theta \approx \dfrac{1}{2} 6 \cdot \sqrt{76} \cdot \sin 47^\circ \approx 19.1274$

(**Note:** θ is the angle between the vectors $\angle\,\overrightarrow{CA}, \overrightarrow{CD} = 180^\circ - \angle\,\overrightarrow{AC}, \overrightarrow{CD} \approx 47^\circ$.)

$A_{BCD} = \dfrac{1}{2}\left|\overrightarrow{BC}\right|\left|\overrightarrow{CD}\right| \sin\theta \approx \dfrac{1}{2}\sqrt{56} \cdot \sqrt{76} \cdot \sin 52^\circ \approx 25.7041$

(**Note:** θ is the angle between the vectors $\angle\,\overrightarrow{CB}, \overrightarrow{CD} = 180^\circ - \angle\,\overrightarrow{AC}, \overrightarrow{CD} \approx 52^\circ$.)

Therefore, $A \approx 72.25$.

12 $\overrightarrow{CD} = (2, 6, -6) \Rightarrow \overrightarrow{DC} = (-2, -6, 6)$

$\vec{i} \cdot \overrightarrow{DC} = -2 \Rightarrow \cos^{-1}\left(\dfrac{-2}{\sqrt{76}}\right) \approx 103.3^\circ$

$\vec{j} \cdot \overrightarrow{DC} = -6 \Rightarrow \cos^{-1}\left(\dfrac{-6}{\sqrt{76}}\right) \approx 133.5^\circ$

$\vec{k} \cdot \overrightarrow{DC} = 6 \Rightarrow \cos^{-1}\left(\dfrac{6}{\sqrt{76}}\right) \approx 46.5^\circ$

13 Given $\overrightarrow{AD} = (2,0,-6) \Rightarrow \overrightarrow{DA} = (-2,0,6)$, $\overrightarrow{BD} = (6,0,-4) \Rightarrow \overrightarrow{DB} = (-6,0,4)$,

$\overrightarrow{AC} = (0,-6,0): \left(\overrightarrow{DA} - \overrightarrow{DB}\right) \cdot \overrightarrow{AC} = ((-2,0,6)-(-6,0,4)) \cdot (0,-6,0) = (4,0,2) \cdot (0,-6,0) = 0$

14 $\cos\frac{\pi}{3} = \frac{1}{2}$

$$\cos\theta = \frac{\begin{pmatrix} 3 \\ -k \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ k \end{pmatrix}}{\sqrt{9+k^2+1}\sqrt{1+9+k^2}} = \frac{3+3k-k}{k^2+10}$$

Therefore: $\frac{1}{2} = \frac{3+2k}{k^2+10} \Rightarrow k^2+10 = 6+4k \Rightarrow k^2-4k+4=0 \Rightarrow k=2$

15 $\begin{pmatrix} 2 \\ x \\ y \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 6+x-y=0$

$\begin{pmatrix} 2 \\ x \\ y \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 8-x+2y=0$

Hence, we have to solve the system of equations:

$$\begin{cases} 6+x-y=0 \\ 8-x+2y=0 \end{cases}$$

Adding the equations: $14+y=0 \Rightarrow y=-14$ and $x=-20$

16 Two vectors are parallel if t exists such that $\mathbf{u} = t\mathbf{v}$. Therefore:

$$\begin{pmatrix} 1-x \\ 2x-2 \\ 3+x \end{pmatrix} = t\begin{pmatrix} 2-x \\ 1+x \\ 1+x \end{pmatrix}; \text{ and } \begin{cases} 1-x = t(2-x) \\ 2x-2 = t(1+x) \\ 3+x = t(1+x) \end{cases}$$

From the last two equations, we can see that $3+x = 2x-2 \Rightarrow x=5$; hence, $3+5 = t(1+5) \Rightarrow t = \frac{4}{3}$. Verifying using the first equation, we can see that $x=5$ is the solution.

17 $A\hat{B}C = \angle \overrightarrow{BA}, \overrightarrow{BC}$

So: $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$

$$\cos A\hat{B}C = \frac{\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}}{\sqrt{1+4+9}\sqrt{1+16}} = \frac{1-8+0}{\sqrt{14\cdot 17}} \Rightarrow A\hat{B}C = \cos^{-1}\frac{-7}{\sqrt{14\cdot 17}} \approx 117^\circ$$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{BA} + \overrightarrow{BC} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$

$B\hat{A}C = \angle \overrightarrow{AB}, \overrightarrow{AC}$

$$\cos B\hat{A}C = \frac{\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}0\\6\\3\end{pmatrix}}{\sqrt{1+4+9}\sqrt{36+9}} = \frac{0+12+9}{\sqrt{14\cdot 45}} \Rightarrow B\hat{A}C = \cos^{-1}\frac{21}{\sqrt{14\cdot 45}} \approx 33^\circ$$

18 **a)** $(b, 3, 2)\cdot(1, b, 1) = b + 3b + 2 = 4b + 2 = 0$

Vectors are orthogonal if their dot product is zero; therefore: $4b + 2 = 0 \Rightarrow b = -\frac{1}{2}$

b) $(4, -2, 7)\cdot\left(b^2, b, 0\right) = 4b^2 - 2b + 7\cdot 0 = 4b^2 - 2b$

Vectors are orthogonal if their dot product is zero; therefore: $4b^2 - 2b = 0 \Rightarrow b = 0, \frac{1}{2}$

For $b = 0$, the vector $\left(b^2, b, 0\right)$ is a zero vector. A zero vector has no direction; therefore, it is not orthogonal to any vector. Hence, the vectors are only orthogonal for $b = \frac{1}{2}$.

19 To determine the angle between two vectors, we are going to find their dot product: $(\mathbf{p}+\mathbf{q})(\mathbf{p}-\mathbf{q}) = \mathbf{p}^2 - \mathbf{q}^2$.

Since, for any vector: $\mathbf{v}^2 = |\mathbf{v}||\mathbf{v}|\cos 0^\circ = |\mathbf{v}|^2\cdot 1 = |\mathbf{v}|^2$ we have: $(\mathbf{p}+\mathbf{q})(\mathbf{p}-\mathbf{q}) = \mathbf{p}^2 - \mathbf{q}^2 = |\mathbf{p}|^2 - |\mathbf{q}|^2 = 0$; therefore, the vectors are perpendicular.

20 We can find the z-component by transforming 300 m/min into km/h:

$300 \text{ m/min} = 0.3 \text{ km/min} = 0.3\cdot 60 \text{ km/h} = 18 \text{ km/h}$

The velocity vector in the xy-plane is parallel to the vector $(-1, 1)$. The unit vector in this direction is $\frac{1}{\sqrt{2}}(-1, 1)$. Since the airspeed is 200 km/h, and its vertical component is 18 km/h, then the velocity of the xy-component is: $\sqrt{200^2 - 18^2} = \sqrt{39\,676} = 199.188... \approx 199$ km/h. So, the velocity vector in the xy-plane is: $\frac{\sqrt{39\,676}}{\sqrt{2}}(-1, 1) = \sqrt{19\,838}\,(-1, 1) \approx (-140.8, 140.8)$. Hence, the velocity vector is $(-140.8, 140.8, 18)$.

Exercise 12.3

1 **a)** In the equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, vector $\mathbf{r}_0 = \begin{pmatrix}-1\\0\\2\end{pmatrix}$, and $\mathbf{u} = \begin{pmatrix}1\\5\\-4\end{pmatrix}$, so the vector equation of the line is

$\mathbf{r} = \begin{pmatrix}-1\\0\\2\end{pmatrix} + t\begin{pmatrix}1\\5\\-4\end{pmatrix}$. The parametric equations are $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1+t\\5t\\2-4t\end{pmatrix}$.

b) Substituting $\mathbf{r}_0 = \begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix}2\\5\\-1\end{pmatrix}$ into $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, we get the vector equation of the line:

$\mathbf{r} = \begin{pmatrix}3\\-1\\2\end{pmatrix} + t\begin{pmatrix}2\\5\\-1\end{pmatrix}$. The parametric equations are: $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3+2t\\-1+5t\\2-t\end{pmatrix}$.

c) Substituting $\mathbf{r}_0 = \begin{pmatrix}1\\-2\\6\end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix}3\\5\\-11\end{pmatrix}$ into $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, we get the vector equation of the line:

$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ -11 \end{pmatrix}$. The parametric equations are: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+3t \\ -2+5t \\ 6-11t \end{pmatrix}$.

2 a) In the equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, vector $\mathbf{r}_0 = \mathbf{r}_A = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$, and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 7+1 \\ 5-4 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$, so the vector equation of the line is: $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$.

Note: For $\mathbf{r}_0$ we can use $\mathbf{r}_B$ and for $\mathbf{u}$ we can use $\overrightarrow{BA}$, or any vector parallel to this vector. Therefore, it is possible to find different, but correct, equations of the line.

b) Substituting $\mathbf{r}_0 = \mathbf{r}_A = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 0-4 \\ -2-2 \\ 1+3 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$ into $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, we get the vector equation of the line: $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$.

Note: Since $\mathbf{u}$ is parallel to $\overrightarrow{AB}$, we can use $\mathbf{u} = -\frac{1}{4}\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, and the vector equation of the line would be $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

c) Substituting $\mathbf{r}_0 = \mathbf{r}_A = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 5-1 \\ 1-3 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$ into $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, we get the vector equation of the line: $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$.

3 a) In the equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, we substitute $\mathbf{a} = \mathbf{r}_A = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 5-3 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, so the equation of the line is $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

b) In the equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, we substitute $\mathbf{a} = \mathbf{r}_A = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 5-0 \\ 0+2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, so the equation of the line is $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

4 Method I:
To determine the equation of the line in the required form, we need to find two points on the line $\mathbf{r} = (2,1) + t(3,-2)$. One point is $(2,1)$. Another point we can find by letting, for example, $t = 1$; therefore, the point is $(5,-1)$. The equation of the line through those two points is:

$$\frac{y-1}{x-2} = \frac{-1-1}{5-2} \Rightarrow 3(y-1) = -2(x-2) \Rightarrow 2x + 3y = 7.$$

Method II:

We can write the equation of the line $\mathbf{r} = (2,1) + t(3,-2)$ in parametric form:

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+3t \\ 1-2t \end{pmatrix}$. From the first row: $x = 2 + 3t \Rightarrow t = \frac{x-2}{3}$. Substituting t into the second row, we get:

$y = 1 - 2t = 1 - 2\frac{x-2}{3} \Rightarrow 3(y-1) = -2(x-2) \Rightarrow 2x + 3y = 7$.

Method III:

We can write the equation of the line $\mathbf{r} = (2,1) + t(3,-2)$ in parametric form:

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+3t \\ 1-2t \end{pmatrix}$. Transforming and adding the equations:

$$\begin{cases} x = 2 + 3t \;/ \cdot 2 \\ y = 1 - 2t \;/ \cdot 3 \end{cases}$$

$$+\begin{cases} 2x = 4 + 6t \\ 3y = 3 - 6t \end{cases}$$

$$2x + 3y = 7$$

5 In the equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, the vector $\mathbf{r}_0 = \mathbf{r}_A = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{u}$ can be the same as the direction vector of the given line; therefore, $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$. So: $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j})$.

6 In the equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, the vector $\mathbf{r}_0 = \mathbf{r}_A = (-2, 1, 4)$ and $\mathbf{u} = (3, -4, 7)$. So, we have: $\mathbf{r} = (-2, 1, 4) + t(3, -4, 7)$.

Solution Paper 1 type

7 a The lines are not parallel since the direction vectors (1, 3, 1) and (1, 4, 2) are not a scalar multiple of each other.
For lines to intersect, there should be some point $(x_0, y_0, z_0,)$ which satisfies the equations of both lines, $\mathbf{r} = (2, 2, 3) + t(1, 3, 1)$ and $\mathbf{r} = (2, 3, 4) + s(1, 4, 2)$, for some values of z and s. (Note: We have to change the parameter in one of the equations so that they are not the same.) So:

$x_0 = 2 + t = 2 + s$
$y_0 = 2 + 3t = 3 + 4s$
$z_0 = 3 + t = 4 + 2s$

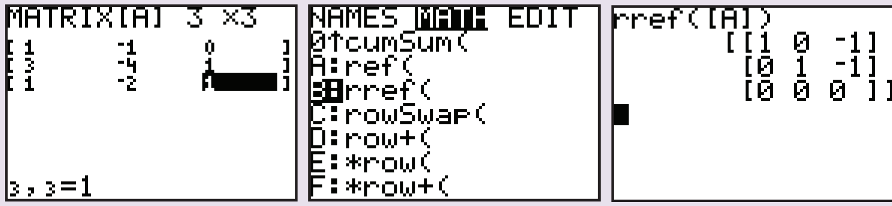
From the first equation, we see that $t = s$. Substituting into the second equation:
$2 + 3t = 3 + 4t \Rightarrow t = -1 \Rightarrow t = s = -1$. Finally, substituting these values into the third equation:
$3 - 1 = 4 - 2 \Rightarrow -2 = -2$. Hence, the lines intersect, and the point of intersection is: $(2, 2, 3) + (-1)(1, 3, 1) = (1, -1, 2)$.

Solution Paper 2 type

7 **a** We can solve this system using matrices. Firstly, transform the system of equations:

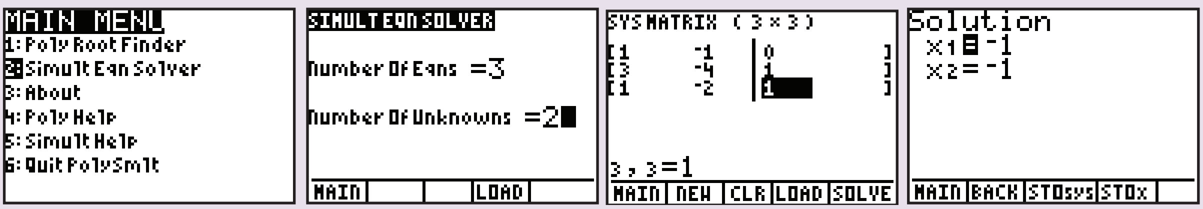
$$\begin{cases} 2+t=2+s \\ 2+3t=3+4s \\ 3+t=4+2s \end{cases} \Rightarrow \begin{cases} t-s=0 \\ 3t-4s=1 \\ t-2s=1 \end{cases}$$

and then use a GDC:



From the result screen, we can see that an intersection exists and that $t = -1$ and $s = -1$.

We can also find the solutions of the system using the Apps menu.



Of course, a GDC will only solve for the parameters – to find the point of intersection, we have to substitute the values of the parameters into the equations of the lines.

Solution Paper 1 type

b The lines are not parallel since the direction vectors (4, 1, 0) and (12, 6, 3) are not a scalar multiple of each other. For lines to intersect, there should be some point (x_0, y_0, z_0) which satisfies the equations of both lines, **r** = (–1, 3, 1) + *t* (4, 1, 0) and **r** = (–13, 1, 2) + *s* (12, 6, 3), for some values of *z* and *s*. (Note: We have to change the parameter in one of the equations so that they are not the same.) So:

$$x_0 = -1+4t = -13+12s$$
$$y_0 = 3+t = 1+6s$$
$$z_0 = 1+0\cdot t = 2+3s$$

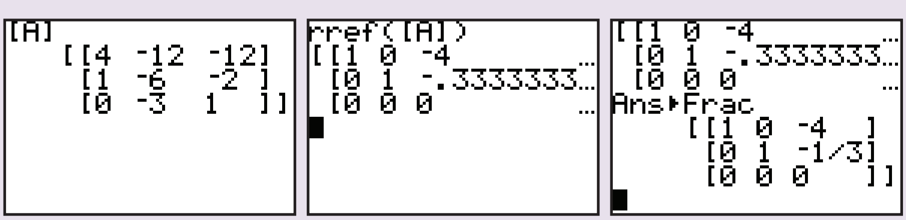
From the last equation, we see that $s = -\frac{1}{3}$. Substituting into the second equation: $3+t = 1+6\left(-\frac{1}{3}\right) \Rightarrow t = -4$.

Finally, substituting these values into the first equation: $-1-16 = -13-4 \Rightarrow -17 = -17$. Hence, the lines intersect, and the point of intersection is: $\mathbf{r} = (-1, 3, 1) + (-4)(4, 1, 0) = (-17, -1, 1)$.

Solution Paper 2 type

b We can solve this system using matrices. Firstly, transform the system of equations:

$$\begin{cases} -1+4t=-13+12s \\ 3+t=1+6s \\ 1+0\cdot t=2+3s \end{cases} \Rightarrow \begin{cases} 4t-12s=-12 \\ t-6s=-2 \\ -3s=1 \end{cases}$$ and then use a GDC:



From the result screen, we can see that an intersection exists, and that $t = -4$ and $s = -\frac{1}{3}$.

Of course, a GDC will only solve for the parameters – to find the point of intersection we have to substitute the values of the parameters into the equations of the lines.

Solution Paper 1 type

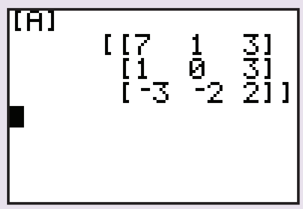
7 c The lines are not parallel since the direction vectors (7, 1, –3) and (–1, 0, 2) are not a scalar multiple of each other. For lines to intersect, there should be some point (x_0, y_0, z_0) which satisfies the equations of both lines, $\mathbf{r} = (1, 3, 5) + t\,(7, 1, -3)$ and $\mathbf{r} = (4, 6, 7) + s\,(-1, 0, 2)$, for some values of z and s. (Note: We have to change the parameter in one of the equations so that they are not the same.) So:

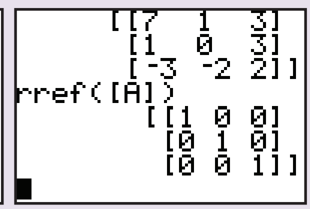
$$x_0 = 1 + 7t = 4 - s$$
$$y_0 = 3 + t = 6 + 0 \cdot s$$
$$z_0 = 5 - 3 \cdot t = 7 + 2s$$

From the second equation, we can see that $t = 3$. Substituting into the first equation: $1 + 21 = 4 - s \Rightarrow s = -18$. Finally, substituting these values into the last equation: $5 - 3 \cdot 3 = 7 + 2(-18) \Rightarrow -4 \neq -29$. Hence, the lines do not intersect; they are skew.

Solution Paper 2 type

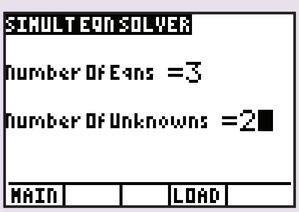
c $\begin{cases} 1 + 7t = 4 - s \\ 3 + t = 6 + 0 \cdot s \\ 5 - 3 \cdot t = 7 + 2s \end{cases} \Rightarrow \begin{cases} 7t + s = 3 \\ t + 0 \cdot s = 3 \\ -3t - 2s = 2 \end{cases}$

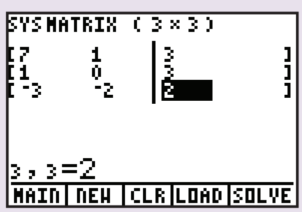


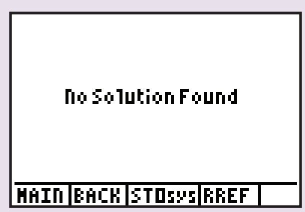


From the result screen, we can see that an intersection does not exist since we interpret the screen as: $t = 0$, $s = 0$ and $0 = 1$.

Alternatively, solving for parameters using Apps:







From the result screen, we can see that there is no solution.

Solution Paper 1 type

d The lines have parallel direction vectors $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$, since $\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$.

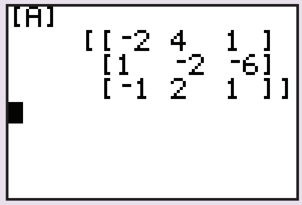
To check whether the lines coincide, we examine the point (3, 4, 6), which is on the first line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, and see whether it lies on the second line, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$.

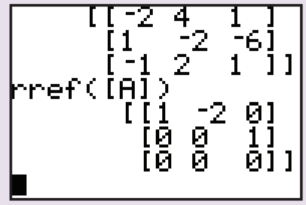
So: $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} 3 = 5 - 4s \Rightarrow s = \frac{1}{2} \\ 4 = -2 + 2s \Rightarrow s = 3 \\ 6 = 7 - 2s \Rightarrow \end{cases}$

We can see that the point is not on the other line, so the lines do not coincide; therefore, the lines are parallel.

Solution Paper 2 type

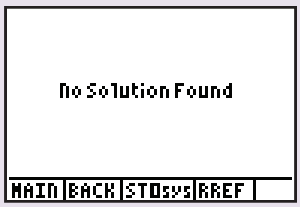
d $\begin{cases} 3-2t=4-4s \\ 4+t=-2+2s \\ 6-1\cdot t=7-2s \end{cases} \Rightarrow \begin{cases} -2t+4s=1 \\ t-2s=-6 \\ -t+2s=1 \end{cases}$





From the result screen, we can see that an intersection does not exist and the lines are parallel. We arrive at this conclusion by interpreting the screen as: $t-2s=0$, $0=1$ and $0=0$.

Alternatively, solving for parameters using Apps:



From the result screen, we can see that there is no solution and that the lines are parallel.

8 **a)** A direction vector is: $(3-2, 2-(-1))=(1,3)$; hence, the equations are:

$\mathbf{r}=(2,-1)+t\,(1,3)$ and $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2+t \\ -1+3t \end{pmatrix}$.

b) We know a point and a direction vector, so the equations are:

$\mathbf{r}=(2,-1)+t\,(-3,7)$ and $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2-3t \\ -1+7t \end{pmatrix}$.

c) For the direction vector, we can use any vector perpendicular to $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$. So, we use vector $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ as the direction vector of the line, since $\begin{pmatrix} -3 \\ 7 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix}=-21+21=0$. Therefore, the equations are: $\mathbf{r}=(2,-1)+t\,(7,3)$ and $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2+7t \\ -1+3t \end{pmatrix}$.

d) We know a point and a direction vector, so the equations are:

$\mathbf{r}=(0,2)+t\,(2,-4)$ and $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2t \\ 2-4t \end{pmatrix}$.

9 **a)** Substituting the point $\left(0,\frac{11}{2},\frac{9}{2}\right)$ into the equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}+t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$:

$$\begin{pmatrix} 0 \\ \frac{11}{2} \\ \frac{9}{2} \end{pmatrix}=\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}+t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} 0=3-2t \\ \frac{11}{2}=4+t \\ \frac{9}{2}=6-t \end{cases}$$

From the first equation, we can see that $t=\frac{3}{2}$. Checking, using the second equation: $\frac{11}{2}=4+\frac{3}{2}\Rightarrow\frac{11}{2}=\frac{11}{2}$, and the third equation: $\frac{9}{2}=6-\frac{3}{2}\Rightarrow\frac{9}{2}=\frac{9}{2}$. So, the point is on the line when $t=\frac{3}{2}$.

b) To check whether the point is on the line, we have to find whether the system of equations has a solution:

$$\begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} -1 = 3 - 2t \\ 4 = 4 + t \\ 6 = 6 - t \end{cases}$$

From the last two equations, we can see that $t = 0$, but this will not satisfy the first equation; hence, there is **no** solution to the system and the point does not lie on the line.

c) We have to solve the system of equations:

$$\begin{pmatrix} \frac{1-2m}{2} \\ \frac{2m}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} \frac{1-2m}{2} = 3 - 2t \\ 2m = 4 + t \\ 3 = 6 - t \end{cases}$$

From the last equation, we can see that $t = 3$. Substituting into the second equation: $2m = 7 \Rightarrow m = \frac{7}{2}$.

Checking, using the first equation: $\frac{1-2\frac{7}{2}}{2} = 3 - 6 \Rightarrow \frac{-6}{2} = -3$. Therefore, the point will be on the line when $m = \frac{7}{2}$.

10 a) i) The starting position is when $t = 0$, so the point is $(3, 4)$.

ii) The velocity vector is $\mathbf{v} = \begin{pmatrix} 7 \\ 24 \end{pmatrix}$.

iii) The speed is $|\mathbf{v}| = \sqrt{7^2 + 24^2} = 25$.

b) i) The starting position is when $t = 0$, so the point is $(-3, 1)$.

ii) The velocity vector is $\mathbf{v} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

iii) The speed is $|\mathbf{v}| = \sqrt{5^2 + (-12)^2} = 13$.

c) i) The starting position is when $t = 0$, so the point is $(5, -2)$.

ii) The velocity vector is $\mathbf{v} = (24, -7)$.

iii) The speed is $|\mathbf{v}| = \sqrt{24^2 + (-7)^2} = 25$.

11 a) The direction of the velocity vector is given by the unit vector: $\frac{1}{\sqrt{(-3)^2 + 4^2}}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

So, the velocity vector is: $160 \cdot \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = 32\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -96 \\ 128 \end{pmatrix}$.

b) The direction of the velocity vector is given by the unit vector: $\frac{1}{\sqrt{12^2 + (-5)^2}}\begin{pmatrix} 12 \\ -5 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix}$.

So, the velocity vector is: $170 \cdot \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{2040}{13} \\ -\frac{850}{13} \end{pmatrix}$.

12 a) The car is travelling from the point (3, 2) to (7, 5), so the direction vector of the velocity vector is given by the unit vector: $\frac{1}{|\mathbf{v}|}\mathbf{v}$, where $\mathbf{v} = (7-3, 5-2) = (4, 3)$. Therefore, the unit vector is: $\frac{1}{\sqrt{4^2+3^2}}(4,3) = \frac{1}{5}(4,3)$; and the velocity vector is: $30 \cdot \frac{1}{5}(4,3) = (24,18)$.

b) The starting point is (3, 2) and the direction vector of the line is (24, 18), so the equation of the position of the car after t hours is $\mathbf{r} = (3, 2) + t\,(24, 28)$.

c) We have to determine the parameter of the point (7, 5).

$$(7,5) = (3,2) + t\,(24,18) \Rightarrow (4,3) = t\,(24,18) \Rightarrow t = \frac{1}{6}$$

Therefore, in $\frac{1}{6}$ of an hour, i.e. 10 minutes, the car will reach the traffic light.

13 a) To be perpendicular to the vectors, both dot products have to be zero.

$$\begin{pmatrix}1\\a\\b\end{pmatrix}\begin{pmatrix}1\\-3\\2\end{pmatrix} = 1-3a+2b = 0 \qquad \begin{pmatrix}1\\a\\b\end{pmatrix}\begin{pmatrix}-2\\1\\-1\end{pmatrix} = -2+a-b = 0$$

So, we have to solve the system: $\begin{cases}-3a+2b = -1\\ \;\;\;a-\;\;b = 2\end{cases} \Rightarrow a = -3, b = -5$

b)
$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} = \frac{\begin{pmatrix}1\\-3\\2\end{pmatrix}\begin{pmatrix}-2\\1\\-1\end{pmatrix}}{\sqrt{1^2+(-3)^2+2^2}\sqrt{(-2)^2+1^2+(-1)^2}} = \frac{-2-3-2}{\sqrt{14}\sqrt{6}}$$

$$= \frac{-7}{2\sqrt{21}} = \frac{-7\sqrt{21}}{2\cdot 21} = -\frac{\sqrt{21}}{6}$$

c) Using the Pythagorean identity for sine, $\sin^2\theta = 1-\cos^2\theta$, and the fact that sine is positive for angles from $0-180^\circ$ we have: $\sin\theta = +\sqrt{1-\cos^2\theta} = \sqrt{1-\frac{21}{36}} = \sqrt{\frac{15}{36}} = \frac{\sqrt{15}}{6}$.

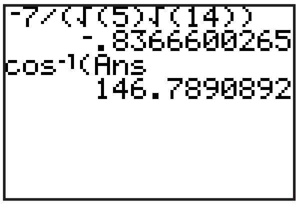
Area of triangle OPQ is: $A = \frac{1}{2}|OP|\,|OQ|\sin P\hat{O}Q = \frac{1}{2}|\mathbf{v}|\,|\mathbf{w}|\sin(\angle\mathbf{v},\mathbf{w})$.

So, $A = \frac{1}{2}\sqrt{14}\sqrt{6}\,\frac{\sqrt{15}}{6} = \frac{1}{2}\,\frac{\sqrt{2\cdot 7}\sqrt{2\cdot 3}\sqrt{3\cdot 5}}{6} = \frac{1}{2}\,\frac{2\sqrt{7}\cdot 3\cdot\sqrt{5}}{6} = \frac{\sqrt{35}}{2}$.

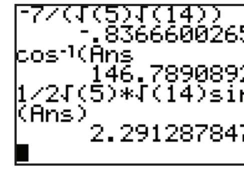
14 a) Firstly, we have to determine vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\overrightarrow{AB} = \begin{pmatrix}-1-(-1)\\3-2\\5-3\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix}0-(-1)\\-1-2\\1-3\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}0\\1\\2\end{pmatrix}\begin{pmatrix}1\\-3\\-2\end{pmatrix}}{\sqrt{0^2+1^2+2^2}\sqrt{1^2+(-3)^2+(-2)^2}} = \frac{0-3-4}{\sqrt{5}\sqrt{14}} = \frac{-7}{\sqrt{5}\sqrt{14}}$$



Therefore, $\theta = \cos^{-1}\frac{-7}{\sqrt{5}\sqrt{14}} \Rightarrow \theta \approx 147^\circ$



b) The area of the triangle is: $A = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| \sin\theta = \frac{1}{2}\sqrt{5}\sqrt{14}\sin\theta \approx 2.29.$

c) i) Line L_1 goes through the point $(2, -1, 0)$ and its direction vector is $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, so its equation is: $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Line L_2 goes through the point $(-1, 1, 1)$ and its direction vector is $\overrightarrow{AC} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$, so its equation is: $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$.

ii) We have to solve the system of equations:

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} 2 = -1 + s \\ -1 + t = 1 - 3s \\ 2t = 1 - 2s \end{cases}$$

From the first equation, we have $s = 3$, and from the second $t = -7$. Substituting these values into the third equation: $2(-7) = 1 - 2 \cdot 3 \Rightarrow -14 \neq -5$. So, there is no point of intersection.

15 a) Let the direction vector be a vector parallel to $\overrightarrow{AB}$: $\overrightarrow{AB} = \begin{pmatrix} 6-1 \\ -7-3 \\ 8-(-17) \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \\ 25 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$.

Thus, we can use the vector $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ as the direction vector. Therefore, the parametric equations of the line are: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+t \\ 3-2t \\ -17+5t \end{pmatrix}$.

b) If point P is on the line, then vector $\overrightarrow{OP} = \begin{pmatrix} 1+t \\ 3-2t \\ -17+5t \end{pmatrix}$. If $\overrightarrow{OP}$ is perpendicular to the line, then $\overrightarrow{OP}$ and the direction vector of the line are perpendicular, and their dot product is zero.

$$0 = \overrightarrow{OP}\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1+t \\ 3-2t \\ -17+5t \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = 1 + t + -6 + 4t + -85 + 25t = 30t - 90 \Rightarrow t = 3$$

So, $\overrightarrow{OP} = \begin{pmatrix} 1+3 \\ 3-2\cdot 3 \\ -17+5\cdot 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}$ and $P(4, -3, -2)$.